Hybrid/mixed assumed stress element for anisotropic laminated elliptical and parabolic shells

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\textbf{ARTICLE INFO}

Article history:
Received 19 June 2008
Received in revised form 3 June 2009
Accepted 8 June 2009
Available online 7 July 2009

Keywords:
Laminated composite
Orthotropic shell
Elliptical shell
Hybrid/mixed assumed stress element
Shear deformation theory
Assumed strain
Shell theory
Shell element
Finite element

\textbf{ABSTRACT}

A variety of elliptical/parabolic dome type structures are used in important aerospace and civil structural systems such as underwater vehicles, stadium covers, exhibition halls, auditoriums and museum halls. For the analysis of such structures, a shear deformable four-noded finite element based on a hybrid/mixed assumed stress is presented in this paper. The element called iHES (improved Hybrid and Enhanced Shell element) is developed assuming the most general arbitrary orthogonal coordinate system. The element is based on a first-order shear flexible formulation and essentially consists of a combination of drilling degrees of freedom with assumed stress and enhanced strain techniques. Using the element developed here, a detailed parametric study of anisotropic elliptical and parabolic shells of various configurations is carried out to investigate the effects of aspect and height ratios as well as layer lay-up schemes.

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1. Introduction

Non-circular shells arise in dome-type aerospace and civil engineering structures as well as in submersible systems. However, the study of non-circular shells under static and dynamic loading situations has received limited attention in the literature compared to the analyses of conventional circular shells.

One of the earliest works in element formulations for the analysis of elliptical paraboloid shells was reported by Aas\textsuperscript{[1]} who applied it to elliptic paraboloid shells. Later Chetty and Tottenham\textsuperscript{[2]} developed the combined variational method for modeling stresses in hyperbolic paraboloid shells followed by Padilla and Schnobrich\textsuperscript{[3]} who used finite difference to treat elliptic paraboloid shells. Iyengar and Srinivasan\textsuperscript{[4]} suggested the characteristic function method for solving the same problems addressed by Chetty and Tottenham. Further, Swaddiwudhipong\textsuperscript{[5,6]} studied the bending problem of elliptic paraboloid shells. Subsequently, Soldatos\textsuperscript{[7]} and Soldatos and Tzivanidis\textsuperscript{[8]} investigated the free vibrations of multilayered closed oval shells using classical shell theory, whereas Noor\textsuperscript{[9]}, and Kumar and Singh\textsuperscript{[10]} investigated the same shell structures employing the shear deformation theory. Suzuki et al.\textsuperscript{[11]} presented results for non-circular elliptical shells based on classical shell theory. Suzuki et al.\textsuperscript{[12]} later extended their theories to models based on shear deformation theory. Koksal\textsuperscript{[13]} presented the finite differences analysis approach of elliptic, hyperbolic and revolution paraboloid shells with isotropic material. Patel et al.\textsuperscript{[14]}, Sambandam et al.\textsuperscript{[15]} and Ganapathi et al.\textsuperscript{[16]} analyzed buckling and vibration of laminated composite elliptical shells. Recently, Patel et al.\textsuperscript{[17]} presented results investigating the thermo-elastic buckling characteristics of angle-ply laminated elliptical cylindrical shells. Hyer and Pascher\textsuperscript{[18]} introduced variable-thickness elliptical cylinder where the axial load carrying capacity is optimized. The pure bending of oval cylindrical shells using nonlinear formulations emphasizing high deformability of a new class of structures was recently presented by Vazir\textsuperscript{[19]}.

However, the analysis of composite laminated non-circular shells with dome-type configurations appears to be scarce in the literature because of their increased complexity due to various couplings arising from material anisotropy.

The objective of this study is to develop a new hybrid element for anisotropic laminated elliptical and parabolic shells based on first-order shear deformation theory and then use the same element to investigate the structural response of such structures to external loads under varying aspect and height ratios as well as lamination schemes. In particular, for shells subjected to uniformly distributed loads, the discrepancies between the displacements and stress resultants on semi-major and -minor axis are investigated and reported.
2. Theory and formulation

In this paper, the theoretical development for a 6 degrees of freedom four-node finite element for anisotropic elliptical shells is presented. The theoretical development of general shell [20] is based on the following assumptions: (a) linear elastic behavior of laminated anisotropic materials, (b) strain–displacement relations expressed in arbitrary orthogonal curvilinear coordinate system, (c) thin shell theory (i.e., the thickness-direction normal stress is negligible compared with stress tangential to the shell surface) and (d) transverse shear deformation is significant enough to influence the governing equations.

2.1. Strain–displacement relations

Fig. 1 defines the differential element presented in this research along with its coordinate system. The strain–displacement equations are related to the components of the displacement vector as follows [21]:

$$
\begin{align*}
\varepsilon_1 &= \frac{1}{A_1(1 + \sqrt{R_1})} \left( \frac{\partial U_1}{\partial x_1} + \frac{U_2 \partial A_1}{A_2 \partial x_2} + \frac{A_1 W}{R_1} \right) \\
\varepsilon_2 &= \frac{1}{A_2(1 + \sqrt{R_2})} \left( \frac{\partial U_2}{\partial x_2} + \frac{U_1 \partial A_2}{A_1 \partial x_1} + \frac{A_2 W}{R_2} \right) \\
\varepsilon_3 &= \frac{\partial W}{\partial z} \\
\gamma_{1n} &= \frac{\partial W}{A_1 \partial x_1} + \frac{\partial z}{A_1} \left( \frac{U_1}{A_1} \right) + \frac{\partial A_1}{A_2} \frac{\partial W}{\partial x_2} \\
\gamma_{12} &= \frac{\partial W}{A_1 \partial x_1} + \frac{\partial z}{A_1} \left( \frac{U_1}{A_1} \right) + \frac{\partial A_1}{A_2} \frac{\partial W}{\partial x_2} \\
\gamma_{2n} &= \frac{\partial W}{A_2 \partial x_2} + \frac{\partial z}{A_2} \left( \frac{U_2}{A_2} \right) + \frac{\partial A_2}{A_2} \frac{\partial W}{\partial x_1} \\
\gamma_{22} &= \frac{\partial W}{A_2 \partial x_2} + \frac{\partial z}{A_2} \left( \frac{U_2}{A_2} \right) + \frac{\partial A_2}{A_2} \frac{\partial W}{\partial x_1}
\end{align*}
$$

where $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, and $\gamma_{ij}$ are, respectively, the curvilinear coordinates of the surface, the displacement vector components, and the curvature radius and the thickness coordinate, respectively.

$$
\begin{align*}
\tilde{A}_1 &= A_1(1 + \sqrt{R_1}) \\
\tilde{A}_2 &= A_2(1 + \sqrt{R_2})
\end{align*}
$$

Substituting Eq. (2) into Eq. (1) gives

$$
\begin{align*}
\varepsilon_1 &= \frac{\partial U_1}{\partial x_1} + \frac{U_2 \partial A_1}{A_2 \partial x_2} + \frac{A_1 W}{R_1} \\
\varepsilon_2 &= \frac{\partial U_2}{\partial x_2} + \frac{U_1 \partial A_2}{A_1 \partial x_1} + \frac{A_2 W}{R_2} \\
\varepsilon_3 &= \frac{\partial W}{\partial z} \\
\gamma_{1n} &= \frac{\partial W}{A_1 \partial x_1} + \frac{\partial z}{A_1} \left( \frac{U_1}{A_1} \right) + \frac{\partial A_1}{A_2} \frac{\partial W}{\partial x_2} \\
\gamma_{12} &= \frac{\partial W}{A_1 \partial x_1} + \frac{\partial z}{A_1} \left( \frac{U_1}{A_1} \right) + \frac{\partial A_1}{A_2} \frac{\partial W}{\partial x_2} \\
\gamma_{2n} &= \frac{\partial W}{A_2 \partial x_2} + \frac{\partial z}{A_2} \left( \frac{U_2}{A_2} \right) + \frac{\partial A_2}{A_2} \frac{\partial W}{\partial x_1} \\
\gamma_{22} &= \frac{\partial W}{A_2 \partial x_2} + \frac{\partial z}{A_2} \left( \frac{U_2}{A_2} \right) + \frac{\partial A_2}{A_2} \frac{\partial W}{\partial x_1}
\end{align*}
$$

where $\varepsilon^0_1$, $\varepsilon^0_2$, $\kappa$, $\tau_1$, and $\mu^0_1$ are, respectively, the in-plane normal and in-plane shear strains, the change in the curvature and torsion of the reference surface and the shearing strain components ($i, j = 1, 2$).

2.2. Stress resultants and stress couples

Integrating the stress components through the thickness gives the stress resultants and stress couples. These are outlined as

$$
\begin{align*}
N_{11} &= \int_{1}^{2} \tau_{1n} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
N_{12} &= \int_{1}^{2} \tau_{12} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
Q_1 &= \int_{1}^{2} \tau_{1n} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
M_{11} &= \int_{1}^{2} \tau_{12} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
M_{12} &= \int_{1}^{2} \tau_{12} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
N_{22} &= \int_{1}^{2} \tau_{22} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
N_{21} &= \int_{1}^{2} \tau_{21} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
Q_2 &= \int_{1}^{2} \tau_{22} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
M_{22} &= \int_{1}^{2} \tau_{22} \left( 1 + \frac{z}{R_1} \right) d\zeta, \\
M_{21} &= \int_{1}^{2} \tau_{21} \left( 1 + \frac{z}{R_1} \right) d\zeta
\end{align*}
$$

Fig. 1. Differential element and coordinate system of shell. (a) Differential element of a shell; (b) definition of shell coordinate system.
where the quantities \((N_{11}, N_{22}, N_{12}, N_{21})\) are the in-plane stress resultants, \((M_{11}, M_{22}, M_{12}, M_{21})\) are the stress couples resultants and \((Q_1, Q_2)\) are the transverse force resultants.

2.3. Equations of motion

Using the principle of virtual work yields the following relationships:

\[
\frac{\partial^2 N_{11}}{\partial x_1^2} + \frac{\partial^2 N_{22}}{\partial x_2^2} + N_{12} \frac{\partial^2 N_{12}}{\partial x_1 \partial x_2} + N_{21} \frac{\partial^2 N_{21}}{\partial x_2 \partial x_1} = N_1 \frac{\partial^2 u_1}{\partial x_1^2} + N_2 \frac{\partial^2 u_2}{\partial x_2^2}
\]

\[
N_2 \frac{\partial^2 N_{12}}{\partial x_1 \partial x_2} + N_{21} \frac{\partial^2 N_{21}}{\partial x_2 \partial x_1} = N_1 \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + N_2 \frac{\partial^2 u_2}{\partial x_2 \partial x_1}
\]

\[
\frac{\partial^2 N_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 N_{22}}{\partial x_1 \partial x_2} + N_{12} \frac{\partial^2 N_{12}}{\partial x_1^2} + N_{21} \frac{\partial^2 N_{21}}{\partial x_2^2} = N_1 \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + N_2 \frac{\partial^2 u_2}{\partial x_2 \partial x_1}
\]

\[
\frac{\partial^2 N_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 N_{22}}{\partial x_1 \partial x_2} + N_{12} \frac{\partial^2 N_{12}}{\partial x_1 \partial x_2} + N_{21} \frac{\partial^2 N_{21}}{\partial x_1 \partial x_2} = N_1 \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + N_2 \frac{\partial^2 u_2}{\partial x_2 \partial x_1}
\]

\[
l_1, l_2, l_3 = \sum_{k=1}^{N} \mu^{k(1)}(\zeta, \zeta^2) d_\zeta
\]

where \(l, \mu^{(k)}\) and \(\zeta\) are, respectively, moments of inertia \((i = 1, 2)\), density of the \(k\)th lamina material and the thickness coordinate. (Fig. 2)

2.4. Constitutive equations

Substituting Eq. (3) into Eq. (4) gives a relationship between the stress resultants and stress couples through the constitutive equations:

\[
\begin{bmatrix}
N_{11} \\
N_{12} \\
N_{22} \\
N_{21}
\end{bmatrix}
= \begin{bmatrix}
G_{ij} & A_{ij} \\
A_{ij} & C_{ij}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \bar{Q}_{ij}}{\partial x_1} \\
\frac{\partial \bar{Q}_{ij}}{\partial x_2}
\end{bmatrix}
+ \begin{bmatrix}
H_{ij} & B_{ij} \\
B_{ij} & H_{ij}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \bar{Q}_{ij}}{\partial x_1} \\
\frac{\partial \bar{Q}_{ij}}{\partial x_2}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
2 \\
2
\end{bmatrix}
\]

\[
i,j = 1, 6, 2, 6
\]

\[
\begin{bmatrix}
M_{11} \\
M_{12} \\
M_{22} \\
M_{21}
\end{bmatrix}
= \begin{bmatrix}
H_{ij} & B_{ij} \\
B_{ij} & H_{ij}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \bar{Q}_{ij}}{\partial x_1} \\
\frac{\partial \bar{Q}_{ij}}{\partial x_2}
\end{bmatrix}
+ \begin{bmatrix}
J_{ij} & D_{ij} \\
D_{ij} & J_{ij}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
2 \\
2
\end{bmatrix}
\]

\[
i,j = 1, 6, 2, 6
\]

\[
\begin{bmatrix}
Q_{11} \\
Q_{12}
\end{bmatrix}
= \begin{bmatrix}
A_{ij} \\
A_{ij}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \bar{Q}_{ij}}{\partial x_1} \\
\frac{\partial \bar{Q}_{ij}}{\partial x_2}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
2 \\
2
\end{bmatrix}
\]

\[
i,j = 1, 6, 2, 6
\]

Fig. 2. Coordinate system of laminated composite. (a) Multi-directional laminate with coordinate notation of individual plies; (b) fiber-reinforced lamina with global and material coordinate systems.

where

\[
\begin{align*}
G_{ij} &= A_{ij} + a_1 B_{ij} + a_2 D_{ij} + a_3 E_{ij}, \\
H_{ij} &= B_{ij} + a_1 D_{ij} + a_2 E_{ij} + a_3 F_{ij}, \\
C_{ij} &= G_{ij} + b_1 B_{ij} + b_2 D_{ij} + b_3 E_{ij}, \\
E_{ij} &= H_{ij} + b_1 D_{ij} + b_2 E_{ij} + b_3 F_{ij}, \\
J_{ij} &= D_{ij} + a_1 E_{ij} + a_2 F_{ij} + a_3 C_{ij}, \\
J_{ij} &= D_{ij} + b_1 E_{ij} + b_2 F_{ij} + b_3 C_{ij}
\end{align*}
\]
shear force resultants (thickness. This discrepancy between the actual stress state and the plates and shells, the transverse shear stresses vary thought the layer 3. Hybrid/mixed stress element is rank sufficient and invariant and is derived using a mixed with assumed stress and enhanced strain techniques[24]. This ele-

\[ A_{55} = a_1 B_{55} + a_2 D_{55} + a_3 E_{55} \]
\[ \begin{bmatrix} \mathbf{b} \\ 44 \end{bmatrix} = A_{44} + b_1 B_{44} + b_2 D_{44} + b_3 E_{44} \]
\[ A_{ij} = \frac{1}{N} \sum_{k=1}^{N} \left( \bar{Q}_{ib} \right) \left( h_k - h_{k-1} \right) \]
\[ a, \beta = 4.5 \]
\[ D_{ij} = \frac{1}{N} \sum_{k=1}^{N} \left( \bar{Q}_{ib} \right) \left( h_k^2 - h_{k-1}^2 \right) \]
\[ E_{ij} = \frac{1}{3} \sum_{k=1}^{N} \left( \bar{Q}_{ib} \right) \left( h_k^3 - h_{k-1}^3 \right) \]

Here, \( h_k \) is the thickness of the \( k \)th layer. In composite laminated plates and shells, the transverse shear stresses vary thought the layer thickness. This discrepancy between the actual stress state and the constant stress state is often corrected in computing the transverse shear force resultants \( (Q_1, Q_2) \) with a parameter \( K_s \) commonly called the shear correction factor.

Finally,

\[ \begin{bmatrix} N_{11} \\ N_{12} \\ Q_{11} \\ Q_{12} \\ Q_{21} \\ Q_{22} \\ M_{11} \\ M_{12} \\ M_{21} \\ M_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & G_{16} & 0 & A_{12} & A_{16} & 0 & H_{11} & H_{16} & B_{12} & B_{16} \\ G_{16} & G_{66} & 0 & A_{26} & A_{66} & 0 & H_{21} & H_{61} & B_{21} & B_{61} \\ 0 & 0 & K_{A_{55}} & 0 & 0 & K_{A_{44}} & 0 & 0 & 0 & 0 \\ A_{11} & A_{26} & 0 & G_{22} & G_{62} & 0 & B_{11} & B_{26} & H_{12} & H_{26} \\ A_{16} & A_{66} & 0 & G_{12} & G_{16} & 0 & B_{61} & B_{66} & H_{62} & H_{66} \\ 0 & 0 & 0 & K_{A_{44}} & 0 & 0 & K_{A_{55}} & 0 & 0 & 0 \\ H_{11} & H_{16} & 0 & B_{12} & B_{16} & 0 & J_{11} & J_{16} & D_{12} & D_{16} \\ H_{21} & H_{61} & 0 & B_{26} & B_{61} & 0 & J_{21} & J_{61} & D_{26} & D_{61} \\ B_{11} & B_{26} & 0 & H_{12} & H_{62} & 0 & D_{11} & D_{16} & J_{12} & J_{62} \\ B_{61} & B_{66} & 0 & H_{62} & H_{66} & 0 & D_{61} & D_{66} & J_{62} & J_{66} \end{bmatrix} \]

The \( \epsilon_1^0, \gamma_1^0, \ldots \) and \( \tau_2 \) were given earlier in Eq. (3) and the \( A_{ij}, B_{ij}, \ldots \) are defined by Eq. (6).

3. Hybrid/mixed stress element

Here, an accurate hybrid/mixed assumed stress four-node shell finite element for the analysis of composite shell structures is presented. The element is based on first-order shear flexible formulation and consists of a combination of drilling degrees of freedom [22,23] with assumed stress and enhanced strain techniques [24]. This element is rank sufficient and invariant and is derived using a mixed formulation. The elements are derived using the unified formulation presented by Di and Ramm [25], while the stress interpolation matrices are developed using the stress mode classification method presented by Feng et al. [26].

The stress field for the in-plane part is assumed to be

\[ \sigma = \sigma_t + \sigma_h = p \bar{N} \] \( \bar{N} = k \bar{A} + p \bar{B} \)

\[ \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \]

\[ \begin{bmatrix} \eta & 0 & -\xi & 0 & \xi^2 & 0 & 2\xi \eta & 0 & 0 & 0 \\ 0 & \xi & 0 & -\eta & 0 & \eta^2 & 0 & 2\eta \xi & 0 & 0 \\ 0 & 0 & \eta & \xi & 0 & 0 & -\eta^2 & -\xi^2 & \xi \eta \end{bmatrix} \begin{bmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} \]

The possible incompatible displacement modes, \( N^{enh}_k \), to be added to a four-node quadrilateral element are shown in Fig. 3.

The resulting expressions for enhanced strain components can be expressed by

\[ \varepsilon^{enh} = \begin{bmatrix} \varepsilon^{enh}_1 \varepsilon^{enh}_2 \varepsilon^{enh}_3 \end{bmatrix} = B^{enh} \lambda \]

Fig. 3. Incompatible displacement modes: \( N_1^{enh} = 1 - \xi^2 \), \( N_2^{enh} = 1 - \eta^2 \).

4. Numerical results and discussions

To establish the accuracy of the element derived here, a number of established test cases taken from the literature were solved and compared with reported results using a custom computer program developed based on the above formulations. Subsequently, we solved a number of real-life orthotropic and laminated elliptical and parabolic shells to demonstrate the applicability of the element for analyzing real-life structures.

4.1. Numerical verification of iHES

The present finite element iHES (Improved Hybrid and Enhanced Shell element) is used for the analysis of a variety of numerical
problems consisting of isotropic and laminated composite shells. Established test cases in the literature are taken for comparison to demonstrate the efficiency and accuracy of the shell element developed here. Wherever appropriate, the results presented here are normalized with the analytical solution. A list of elements used for comparison with the proposed elements is outlined in Table 1.

### 4.1. Pinched cylinder

A pinched cylinder with rigid end diaphragms subjected to two pinching forces is considered (Fig. 4). This is one of the most severe tests of an element’s ability to model both inextensional bending and complex membrane states. Belytschko et al. [32] pointed out the difficulty in passing this test. The normalized radial displacement at the location of the point load is 0.00018248. As shown in Fig. 5, the present element, $iHES$, is noted to give excellent results when compared with results obtained from the literature (Fig. 5).

### 4.1.2. Scordelis-Lo roof

The Scordelis-Lo roof is a singly curved shell structure (see Fig. 6) supported by rigid diaphragms at the edges. It is loaded by gravity force alone making it a membrane-dominated problem. The parameter of interest is the vertical displacement at the mid-point of the free edge (point A in Fig. 6). Due to symmetry of the structure and the loading, only one quadrant of the shell is considered in the analysis of the problem. With only a $10 \times 10$ mesh, good convergence to the reference solution is obtained. The normalized values for vertical displacement at the free edge are shown in Table 2. The normalized reference solution is 0.3024. The $iHES$ element shows a rapid and monotonic convergence.

### 4.1.3. Hemispherical shell

Two geometries are used for this problem; one is a full hemispherical shell (see Fig. 7(b)) and another is a hemispherical shell with $18^\circ$ hole (see Fig. 7(a)). Both shells have the same radius, thickness, material properties and loading conditions. The loading consists of two pairs of inward and outward point loads applied at $90^\circ$ apart. Two essential properties must be demonstrated by the new shell element $iHES$ to validate its superior performance. These properties that need to be well captured by this element are inextensional bending mode and a rigid body motion. Typically, elements with a membrane locking problem cannot solve this example correctly. In this example, the problem is modeled using only one quarter of the hemisphere.

The normalized displacement of the hemispherical shell with a hole is 0.093 [33], whereas the value of the normalized displacement at the loading point as given by MacNeal and Harder (1985) is 0.094.
Therefore, the exact value should be somewhere between these two values. As shown in Table 3(a), the results obtained by iHES compare well with the results reported in the literature.

A different mesh topology is sometimes employed to analyze the full hemispherical shell problem. The geometry and discretization are depicted in Fig. 7(b), where the chosen discretization implies that quadrilateral shell elements become highly warped. An exact analytical solution of 0.0924 is presented by Parsch [24], while numerical results are presented in tabulated form in Table 3(b).

The results of the present shell element are remarkably better than the references used for comparison. It is worth mentioning, however, that even though the element’s ability to address rigid body motion and inextensional bending modes well is demonstrated here, the solution converges very slowly without the implementation of the warping correction introduced by Taylor [34].

### 4.2. Numerical examples for iHES

Once the accuracy of the element is established through analysis of benchmark test cases as given in Section 4.1, we, then, solve a number of numerical examples to further demonstrate its robustness and applicability to real-life elliptical and parabolic shells. A laminated composite elliptical and parabolic shell is considered with the coordinates \( x \) (or \( \phi \)) along the hoop direction, \( y \) (or \( \theta \)) along the meridian direction and \( z \) along the thickness direction having origin at the mid-plane of the shell, as shown in Figs. 8 and 9. The material properties used are summarized in Table 4 where \( E \), \( G \) and \( v \) are the Young’s modulus, shear modulus and Poisson’s ratio of the material, respectively.

The radius of curvature of the middle surfaces for elliptical shell is described as \( R = (b^2/R_0)(1 + \mu_0 \cos 2\psi)^{3/2} \) where \( R_0 = [(a^2 + b^2)/2]^{1/2} \) is the representative radius, \( \psi \) is a variable that denotes an angle between the tangent at the origin of \( x \) (hoop coordinate) and the one at any point on the center-line, and \( \mu_0 = (a^2 - b^2)/(a^2 + b^2) \). The parameters \( a \), \( b \) and \( h \) are semi-major, semi-minor and \( Z \)-axis, respectively. The shell is assumed to be loaded uniformly in the global \( Z \)-axis direction.

All the layers are of equal thickness and ply-angle (\( \beta \)) is measured with respect to the \( x \)-axis (hoop axis). The following non-dimensional forms are used to represent results in graphical and tabular forms:

**Displacement**: \( \tilde{U}_{K,Y,Z,R} = U_{K,Y,Z,R} \left( \frac{E_f h^3}{b^4 q_0} \right) \)

**Stress resultant (Force)**: \( \tilde{N}_{K,Y,Z,R} = N_{K,Y,Z,R} \left( \frac{1}{q_0} \right) \)

\[ \tilde{M}_{K,Y,Z,R} = M_{K,Y,Z,R} \left( \frac{1}{b^4 q_0} \right) \]

---

**Table 3**

<table>
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<tr>
<th>Mesh</th>
<th>Simo et al. [33]</th>
<th>NMS-4F</th>
<th>QUAD4</th>
<th>Kim et al. [34]</th>
<th>QCSD-SA</th>
<th>iHES</th>
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<td>1.038</td>
<td>-</td>
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</tr>
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<td>0.996</td>
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**Table 4**

<table>
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<th>Mesh</th>
<th>Simo et al. [33]</th>
<th>QPH</th>
<th>MITC4</th>
<th>Kim et al. [34]</th>
<th>QCSD-SA</th>
<th>iHES</th>
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**Fig. 7.** (a) Hemispherical shell with 18° hole and (b) full hemispherical shell.
Elliptical shells with orthotropic laminates are analyzed by varying the aspect and height ratios. Displacements and stress resultants are investigated with variations in aspect and height ratios. The aspect ratio, \( \gamma \), of elliptical shell is taken here to mean the ratio of the radius length of \( X \)-axis to that of \( Y \)-axis. On the other hand, the height ratio, \( \lambda \), of elliptical shell is taken as the radius length of \( Z \)-axis divide into that of \( Y \)-axis.

### 4.2.1. Influences of aspect ratio

The effect of aspect ratio \( a/b \) of elliptical shells is analyzed in this section. By varying the aspect ratio (\( a/b = 1.0, 1.5, 2.0, 2.5 \) and \( 3.0 \)) of elliptical shells subjected to vertical uniformly distributed load, the displacement and stress resultant characteristics are investigated for different values of height ratio (i.e., \( h/b = 0.25, 0.50, 0.75 \) and \( 1.0 \)).

First, the vertical displacement \( U_Z \) of vertex point in the elliptical shell is investigated. Table 5 and Fig. 10 summarize the variation

<table>
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<th>Height ratio, ( h/b )</th>
<th>Aspect ratio, ( a/b )</th>
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<tr>
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<tr>
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<td>0.75</td>
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<tr>
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<td>1.0 2.0 3.0 4.0 5.0 6.0 7.0</td>
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**Table 5**

Maximum vertical deflection in elliptical shells for various aspect ratio and height ratio values.

**Fig. 10.** Effect of aspect ratio (\( a/b \)) and height ratio (\( h/b \)) on maximum vertical displacement \( U_Z \).
of the nondimensionalized maximum vertical displacements versus aspect ratio $a/b$ of elliptical shells. The vertical deflection at the vertex point of shells is observed to converge to a constant value for $a/b \leq 4.0$. Further, it is noted that the behavior of elliptical shells is sensitive between aspect ratios of 1.0 and 3.0.

Fig. 11 shows plots of nondimensionalized global displacement $RSLT$ (i.e., $(U_{X}^2+U_{Y}^2+U_{Z}^2)^{1/2}$) along $X$- and $Y$-axis of elliptical shells for various aspect ratios. The maximum deflection is observed at the vertex point of shell. But the displacement $RSLT$ of the boundary edge in $Y$-axis is half as large as the maximum displacement at the vertex point. The increase in the height ratio $h/b$ gives rise to this displacement. Therefore, elliptical shells with high values of aspect ratio and height ratio must be reinforcend at the boundary edge of the small radius axis (i.e., minor axis) to reduce displacements.

Next, Figs. 9–12 show the variation of the nondimensionalized stress resultants with aspect ratio at points ‘P5’ and ‘P6’ on the meridian axis (as defined in Fig. 12). Figs. 13 and 14 show the distribution of in-plane stress resultants while Figs. 15 and 16 show the moment stress resultants variations.

For aspect ratio $a/b > 2.5$, the in-plane forces, $N_p$, adjacent to the vertex point on $Y$-axis tend to initially increase in a somewhat steep manner and then converge. On the other hand, as shown in Fig. 15a, with an increase in aspect ratio, the ‘hoop’ moment forces, $M_p$, adjacent to the vertex on $X$-axis (i.e., ‘P1’) increase much more significantly than the same moments at point ‘P2’ along the $Y$-axis. At points away from the vertex (i.e., points ‘P5’ and ‘P6’), however, the hoop moments behave exactly opposite; with hoop moments along $X$-axis decreasing while the corresponding moments...
served with meridian moments along near the edges (i.e., points \( P5 \) and \( P6 \)), an interesting contrast is observed with meridian moments along Y-axis increasing much more rapidly than those in the X-axis when aspect ratio increases (see Figs. 16a–c).

Further, with an increase in the radius of X-axis, the stiffness of the same direction does not seem to have a correspondingly significant effect on the total structural systems. This suggests that the behavior along the minor axis (or Y-axis) is the critical design factor and, therefore, such shells need to be reinforced at weak points in the same direction does not seem to have a correspondingly significant effect on the total structural systems. This suggests that the stress resultant tends to gradually converge with increase in aspect ratio. Figs. 13–16 are helpful in a design process as the coefficient of stress resultants corresponding to any given aspect ratio \( \alpha/\beta \) and height ratio of \( h/b \) can be directly read from these charts.

Next, deformed shapes and stress resultant contours of elliptical shells are presented. Figs. 17–19 show the deformed shapes for elliptical shells with aspect ratios 1.0 and 2.0, respectively. As shown in Figs. 18 and 19, deformations happen largely near the boundary on X-axis. The maximum displacement is observed to be at the vertex point of elliptical shells. It is known that the behavior of elliptical shells is largely influenced by Y-axis geometry, especially as aspect ratio \( \alpha/\beta \) increases. This is expected as the curvature directly influences stiffness significantly.

4.2.2. Influences of height ratio (height/chord ratio)

In the previous section, the effect of aspect ratio of orthotropic laminated elliptical shells was discussed. In this section, the effect of height ratio (height/chord ratio) \( h/b \) is investigated in detail. By varying the height ratio \( h/b = 0.5–3.0 \), the displacement and stress resultant characteristics of elliptical shells are closely examined.

Fig. 20 shows plots of the variations of nondimensionalized vertical displacement with height ratio \( h/b \). The figure suggests that as the height ratio \( h/b \) increases, the vertical deflection of the shell decreases for all aspect ratios \( \alpha/\beta = 1.00–2.00 \) and then increases again. In Fig. 16, the dotted line represents the minimum deflection points.

Next, Figs. 21–24 show the variation of the nondimensionalized stress resultants with height ratio \( h/b \) at specific points on the meridian axis (i.e., ‘P1’–‘P6’ defined earlier in Fig. 12). Figs. 21 and 22 show the variation of in-plane stress resultants while Figs. 23 and 24 show the variation of moment force resultants with height/chord ratio.

In Figs. 21–24, the solid lines indicate numerical results at specific point on Y-axis while the dashed lines correspond to those on X-axis. The stress resultants along X-axis are almost constant independent of the aspect ratio. In particular, the moment forces on X-axis for elliptical shells with \( \alpha/\beta = 1.0, 1.25 \) and 1.50 are in very close agreement with those on Y-axis for \( \alpha/\beta = 1.0 \). As the height ratio increases, the in-plane forces linearly increase while the...
moment forces decrease in a quadratic fashion. Therefore, it is noted that the aspect and height ratios greatly affect the moment and in-plane forces, respectively.

4.2.3. Laminated elliptical and parabolic shell

In this section, displacements of composite laminated (0/90/90/0) elliptical and parabolic shells under uniformly distributed loads are
investigated. The effect of aspect ratio as well as the effective radius length $R_0$, which is defined above, is examined closely. Fig. 25 shows Model I (parabolic shell) and Model II (elliptical shell) with their corresponding geometrical shape function along global Z-axis. Figs. 26–28 show the displacement patterns for parabolic and elliptical shell with $h/b = 0.5$. As demonstrated by these figures, elliptical

![Fig. 18. Deformed shape of elliptical shell with $h/b = 1.0$. (a) $a/b = 1.0$; (b) $a/b = 2.0$.](image1)

![Fig. 19. Deformed shape of elliptical shell with $h/b = 2.0$. (a) $a/b = 1.0$; (b) $a/b = 2.0$.](image2)

![Fig. 20. Variation of maximum deflection $U_z$ with height ratio ($h/b$).](image3)

![Fig. 21. Variation of hoop in-plane forces $N_0$ with height ratio ($h/b$) at points 'P1' and 'P2'.](image4)
Figure 22. Variation of meridian in-plane forces $N_{p}$ with height ratio $(h/b)$ at points 'P5' and 'P6'.

Figure 23. Variation of hoop moment forces $M_{p}$ with height ratio $(h/b)$ at points 'P1' and 'P2'.

shells differ from the parabolic shell on the location of the point of maximum deflection.

For elliptical shells, maximum displacement occurs – as expected – at the vertex. For parabolic shells, however, the maximum displacement happens to be located near the boundary as shown in Fig. 28. As the aspect ratio increases, the vertical deflection at vertex point of parabolic shell increases. In general, displacements in elliptical shells are higher than those in parabolic shells. Further, Fig. 28 demonstrates that elliptical and parabolic geometries significantly affect the response of non-circular shells.

4.2.4. Elliptical shell with cross-ply laminates

The previous section covered a discussion of the special case of symmetric laminates. In this section, cross-ply laminates of special lamination schemes are considered for elliptical shells. Table 6 summarizes the extensional and bending stiﬀnesses of a variety of symmetric cross-ply laminates.

In this section, the response of elliptical shell is investigated first for the special case of one-direction reinforcement of fiber (i.e., (0/0/0/0) and (90/90/90/90)). Here, the lamination schemes (0/0/0/0) and (90/90/90/90) are parallel to hoop and meridian axis of shells, respectively. Figs. 29–31 show that the displacements of each direction except the displacement $U_{X}$ on the global axis are the smallest when the stacking sequence of laminate is (90/90/90/90). For aspect ratio $a/b = 4$, the maximum displacement $U_{X}$ of the elliptical shell with (0/0/0/0) is about 2.0 times greater than that of the shell with (90/90/90/90) as shown in Fig. 32. For aspect ratio of $a/b \leq 2.5$, the maximum displacement $U_{X}$ of the elliptical shell with (90/90/90/90) is observed to be larger than that of the shell with (0/0/0/0) lay-up scheme. But, for aspect ratio $a/b > 2.5$, the stacking scheme with the maximum displacement $U_{X}$ are (0/90/90/0) and (0/0/0/0). As shown in Figs. 30–32 the displacements tend to have the smallest values for the most part when the elliptical shell is reinforced with (90/90/90/90) or (90/0/0/90) lamination scheme.

The magnitudes of displacements seem to correspond directly with the bending stiffness $D_{zz}$ given in Table 6. The (90/0/0/90) and (90/90/90/90) schemes have the highest value for the bending stiffness $D_{zz}$ ('zz' indicates the meridian axis of the elliptical shell). Therefore, the structural ability of the elliptical shell can be increased by the additional reinforcement in direction to meridian of the elliptical shell.

5. Conclusions

A newly developed shear deformable hybrid element with mixed assumed stress and enhanced strain formulations that can be used to model generally anisotropic laminated elliptical and parabolic shells is presented here. General equations of multilayered laminated anisotropic shells are developed by taking into account shear deformation. The derivation used geometrically linear theory for small elastic strains and strains expressed in orthogonal curvilinear coordinates relevant for generalized shells. The virtual work principle was applied in order to derive the equilibrium equations. To improve the accuracy of the analysis, hybrid/mixed assumed stress shell element is formulated. The FE implementation is cast into a four-noded quadrilateral shell element with 6 degrees of freedom per node. The presented shell elements provide very good results to most problems when compared to other four-node shell elements reported in the literature.

Detailed parametric studies were carried out to bring out the effects of aspect ratio, height ratio, lay-up and cross- and angle-ply on the static behavior characteristic of elliptical shells. Based on the above theoretical developments and numerical results, the following concluding remarks are made:

1. The vertical deflection at the vertex point of elliptical shell and the maximum displacement $U_{X}$ converge to a constant for $a/b > 6.0$. A tendency to increase up to an aspect ratio of $a/b = 3.0$ and then decrease for aspect ratio larger than 3.0 is observed for the maximum displacement $U_{X}$. Consequently, it is noted that the behavior of the elliptic shell is sensitive between aspect ratios 1.0 and 3.0.
2. The increase in radius in X-axis has minor effect in the stiffness in that direction as well as the response of the overall
Fig. 24. (a) Variation of meridian moment forces $M_{\phi}$ with height ratio ($h/b$) at points ‘P1’ and ‘P2’. (b) Variation of meridian moment forces $M_{\phi}$ with height ratio ($h/b$) at points ‘P3’ and ‘P4’. (c) Variation of meridian moment forces $M_{\phi}$ with height ratio ($h/b$) at points ‘P5’ and ‘P6’.

(3) As the height ratio increases, the in-plane forces linearly increase and the moment forces decrease in a quadratic fashion. The deflection of elliptical shells exhibits a peculiar variation with aspect ratio where it decreases first and then increases again.

(4) For the design of elliptical shells, the variation of the coefficient of stress resultants with the aspect ratio $a/b$ and height ratio $h/b$ is presented in figures and tables in a convenient fashion that can be directly used in design practice.

(5) Displacements are smallest when the laminate sequences are [90/90/90/90] or [90/0/0/90]. The given laminate has the highest value for the bending stiffness in the direction of the meridian axis. The structural ability of elliptical shell can be increased by the additional reinforcement in direction to meridian of the elliptical shell. But, the stress resultants can be reduced by reinforcement in the direction of not only the meridian axis $\phi$ but also in the hoop direction, $\theta$. 
Fig. 25. The geometrical shape of the parabolic/elliptical shell. (a) Parabolic shell; (b) elliptical shell.

Fig. 26. (a) Effect of aspect ratio \(a/b\) on \(U_z\) along \(Y\)-axis of parabolic shell \((h/R = 0.5)\). (b) Effect of aspect ratio \(a/b\) on \(U_z\) along \(Y\)-axis of elliptical shell \((h/R = 0.5)\).

Fig. 27. (a) Effect of aspect ratio \(a/b\) on RSLT along \(Y\)-axis of parabolic shell \((h/R = 0.5)\). (b) Effect of aspect ratio \(a/b\) on RSLT along \(Y\)-axis of elliptical shell \((h/R = 0.5)\).

Fig. 28. Deformed shapes of elliptical and parabolic shell. (a) Elliptical shell; (b) parabolic shell.
Table 6
Extensional and bending stiffness of cross-ply elliptical shell.

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Fig. 29. Effect of aspect ratio ($a/b$) and stacking sequence on displacement $U_X$ in elliptical shells.

Fig. 30. Effect of aspect ratio ($a/b$) and stacking sequence on displacement $U_Y$ in elliptical shells.

Fig. 31. Effect of aspect ratio ($a/b$) and stacking sequence on displacement $U_Z$ in elliptical shells.

Fig. 32. (a) Effect of stacking sequence on displacement $RSLT$ along $X$-axis. (b) Effect of stacking sequence on displacement $RSLT$ along $Y$-axis.
In general, realistic dome-type shell structures are loaded by not only their self-weight but also wind load. Extension of this work to dynamic analysis considering lateral loads such as wind load is currently being pursued.

References